QFT I - Problem Set 1

(1) Self Interaction

For the one dimensional solid state lattice *without* nearest neighbor interaction, the Hamiltonian is

$$H = \sum_j \left\{ \frac{D}{2} Q_j Q_j + \frac{1}{2M} P_j P_j \right\}.$$

We would like to add a so called self-interaction to this Hamiltonian, that is an interaction not with any neighbor but at the same lattice site j. Later in the lecture, you will see that self interactions of quartic powers are particularly important. So we are talking of the Hamiltonian

$$H = \sum_j \left\{ \frac{D}{2} Q_j Q_j + \frac{1}{2M} P_j P_j + \lambda\, Q_j Q_j Q_j Q_j \right\},$$

and in particular of the self-interaction piece

$$H_{\text{self in.}} = \sum_j \lambda\, Q_j Q_j Q_j Q_j .$$

Now, given the expression of Q_j in terms of creation and annihilation operators

$$Q_j = \frac{1}{\sqrt{2}\,(DM)^{1/4}} \left(a_j + a_j^\dagger \right),$$

and the Fourier decomposition of a and $a^\dagger$,

$$a_j = \frac{1}{\sqrt{\mathcal{N}}} \sum_q e^{iaqj} a_q, \qquad a_j^\dagger = \frac{1}{\sqrt{\mathcal{N}}} \sum_q e^{-iaqj} a_q^\dagger,$$

express $H_{\text{self. in.}}$ in terms of a and $a^\dagger$ in Fourier space. You do not need to multiply out the $(a + a^\dagger)^4$ pieces, but if you like, go ahead and convince yourself that you end up with quite some terms.
Hints: For our periodic lattice $\frac{1}{\mathcal{N}} \sum_j e^{iajq} = \delta_{0,q}$, where q can be the sum of momenta, i.e. $q = q_1 + q_2 + q_3 + \ldots$ in general. Which brings us to the final hint: each power of Q_j needs its own Fourier decomposition.

(2) One-Phonon State

Compute the mean square displacement for a one-phonon state in position space,

$$\langle j | Q_j^2 | j \rangle$$

for
(a) an uncoupled phonon state,
(b) for a state with nearest-neighbor interactions.
Hints: Express Q_j in terms of creation and annihilation operators $a_j^\dagger, a_j$. For (b), figure out how the latter relate to the physical creation and annihilation operators $A_j^\dagger, A_j$.