QFT I - Problem Set 8

(15) Your first own perturbation result

In problem (14), you developed the important relation

$$Z[j^*, j] = \exp\left[iS_I[i\frac{\delta}{\delta j}, i\frac{\delta}{\delta j^*}]\right] Z_0[j^*, j]$$

for the partition function

$$Z[j^*, j] = \int \mathcal{D}(\phi^*, \phi) \exp\Big\{iS[\phi^*, \phi] - i\int_x (j(x)\phi^*(x) + j^*(x)\phi(x))\Big\}, \tag{1}$$

where $x = (t, \boldsymbol{x})$. For the Lagrangian of (14), you did also show that the non-interacting part $Z_0[j^*, j]$ can be written as

$$Z_0[j^*, j] = \mathcal{N} \exp\left(\int_p j^*(p)\frac{i}{\omega - \boldsymbol{p}^2/(2M)}j(p)\right),$$

where $p = (\omega, \boldsymbol{p})$ and the normalization factor $\mathcal{N}$ was actually $\det[\omega - \boldsymbol{p}^2/(2M)]^{-1}$, but that will be unimportant in the following. In fact, as far as correlation functions such as the propagator are concerned, $\mathcal{N}$ drops out by normalizing the partition function just like in the lecture as

$$Z_{norm.}[j^*, j] = \frac{\int \mathcal{D}(\phi^*, \phi) \exp\Big\{iS[\phi^*, \phi] - i\int(j\phi^* + j^*\phi)\Big\}}{\int \mathcal{D}(\phi^*, \phi) \exp\Big\{iS[\phi^*, \phi] - i\int(j\phi^* + j^*\phi)\Big\}_{|j,j^*=0}} = \frac{Z[j^*, j]}{Z[0]}. \tag{2}$$

In the following problem, we will compute $Z_{norm.}[j^*, j]$ to first order in the interaction and from this infer the propagator to first order in perturbation theory. You are invited to try this first without the detailed step-by-step problems given below. The interacting piece of the action was

$$S_I = \int_x \frac{\lambda}{2}\big(\phi^*(t, \boldsymbol{x})\phi(t, \boldsymbol{x})\big)^2.$$

The results can be obtained either in position or momentum space. As one usually works in momentum space, we want to perform the calculation in momentum space. So ...

a) Fourier transform S_I to Fourier space.
You should find that $S_I = \frac{\lambda}{2}\int_{p_1}\int_{p_2}\int_{p_3}\int_{p_4} \delta(p_1 - p_2 + p_3 - p_4)\phi^*(p_1)\phi(p_2)\phi^*(p_3)\phi(p_4)$.

b) To compute the numerator of Eqn. (2) ...

b1) Expand $\exp(iS_I)$ to first order in λ and substitute $\phi(p) \to i\frac{\delta}{\delta j^*(p)}$, $\phi^*(p) \to i\frac{\delta}{\delta j(p)}$

b2) Concentrate on the term proportional to λ: Ignore for a moment the $\frac{i\lambda}{2}\int_{p_1 \dots p_4} \delta(p_1 - p_2 + p_3 - p_4)$ part of this term and compute

$$\frac{\delta}{\delta j(p_1)}\frac{\delta}{\delta j^*(p_2)}\frac{\delta}{\delta j(p_3)}\frac{\delta}{\delta j^*(p_4)} \exp\left(\int_p j^*(p)K(p)j(p)\right),$$

where we have abbreviated $K(p) \equiv \frac{i}{\omega - \boldsymbol{p}^2/(2M)}$ for convenience.

b3) In total, this will give you 7 terms: One with 4 $j's$, four with 2 $j's$ and two with no $j's$. If you depict each K by a line, λ by a point where four lines meet and identify the number of $j's$ with the number of external lines, you result is roughly speaking

$$\frac{\delta}{\delta j(p_1)}\frac{\delta}{\delta j^*(p_2)}\frac{\delta}{\delta j(p_3)}\frac{\delta}{\delta j^*(p_4)} \exp\left(\int_p j^*(p)K(p)j(p)\right) = [1 \qquad + 4 \qquad + 2 \qquad] \exp\left(\int_p j^*(p)K(p)j(p)\right)$$

b4) Collect all pieces to get an expression for the numerator of (2) to order λ. If you did not do it already: perform the integration over all momenta to get rid of as many δ functions as possible, *except* for the overall momentum conservation $\delta(p_1 - p_2 + p_3 - p_4)$. By changing the names of the momentum variables, you can group the 7 terms in the aforementioned 3 distinct ones.

c) The denominator of (2) is computed like the numerator but it is evaluated at $j = j^* = 0$, so only one of the three diagrams above appears. Convince yourself that to order λ, this term cancels in (2) (essentially using $1/(1+x) = 1 - x + \dots$)

d) The Propagator is given by

$$\langle \phi^*(p)\phi(p')\rangle = \Big(i\frac{\delta}{\delta j(p)}\Big)\Big(i\frac{\delta}{\delta j^*(p')}\Big) Z_{norm}[j^*, j]\Big|_{j=j^*=0}$$

Compute it to first order in λ. To do this ...

d1) Act with $(i\frac{\delta}{\delta j(p)})(i\frac{\delta}{\delta j^*(p')})$ on the expression for $Z_{norm}[j^*, j]$ that we have just computed in (c). Set $j = j^* = 0$. Then perform all momentum integrations.

d2) re-substitute $K(p) = \frac{i}{\omega - \boldsymbol{p}^2/(2M)}$ to appreciate your result in full glory.

e) When you identify each of the $\frac{i}{\omega - \boldsymbol{p}^2/(2M)}$ free propagators with a line, the λ with a point where four lines meet (interact) and an integral over momentum with a corresponding free propagator as a closed loop, convince yourself that diagrammatically, your result is

$$\langle \phi^*(p)\phi(p')\rangle =$$